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SINGULAR MEASURES AS ELEMENTS OF REPRESENTATION SPACES OF COMPLEMENTARY SERIES REPRESENTATIONS OF THE LORENTZ GROUP

A. I. Shtern

Abstract. It is proved that every representation space of the standard realization of any complementary series representation for the Lorentz group contains not only ordinary measurable functions but also some singular measures.

§ 1. Introduction

Let G be the proper Lorentz group treated as the group $\mathrm{SL}(2,\mathbb{C})$, and let V_σ, $0<\sigma<2$, be a representation belonging to complementary series representations of G. Let us recall the corresponding construction ([4], §12). Let $\mathfrak{H}''$ be the space of all functions f on $\mathbb{C}$ which vanish outside sone circle (depending on the function) and are differentiable with respect to x and y $(z=x+iy\in\mathbb{C})$ so many times that the Fourier transforms

$$\varphi(w)=(2\pi)^{-1}\int_{\mathbb{C}} f(z)e^{-i\,\mathrm{Re}(\overline{z}w)}\,dz \tag{1}$$

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are summable functions of $w \in \mathbb{C}$. Denote the set of these Fourier transforms by H''. Let us equip the space $\mathfrak{H}''$ with the inner product $(0 < \sigma < 2)$

$$(f_1, f_2) = \iint |z_1 - z_2|^{-2+\sigma} f_1(z_1)\overline{f_2(z_2)} dz_1 dz_2$$

and equip the space H'' with the corresponding inner product

$$(\varphi_1, \varphi_2) = 2^\sigma \pi \frac{\Gamma(\sigma/2)}{\Gamma(1-\sigma/2)} \int \varphi_1(w)\overline{\varphi_2(w)} |w|^{-\sigma} dw.$$

Then $\mathfrak{H}''$ and H'' become Euclidean spaces; denote their Hilbert space completions by $\mathfrak{H}_\sigma$ and H_σ, respectively.

The Fourier transform (1) turns out to be an isometric mapping of $\mathfrak{H}''$ onto H'' which admits a unique extension by continuity to a unitary operator taking $\mathfrak{H}_\sigma$ onto H_σ.

The space H_σ is the family of all measurable functions φ on $\mathbb{C}$ satisfying the condition

$$\int |\varphi(w)|^2 |w|^{-\sigma} dw < \infty.$$

Thus, the structure of elements of the space H_σ is rather simple.

The objective of this note is to show that, in contrast to H_σ, the structure of elements of $\mathfrak{H}_\sigma$ is much more complicated. Namely, we show that, for every σ, $0 < \sigma < 2$, the space $\mathfrak{H}_\sigma$ contains nontrivial singular measures.

To this end, it suffices to show that the space H_σ contains Fourier transforms of nontrivial continuous singular measures, which is studied in the next section.

§ 2. Main theorem

The following result of Hewitt and Ritter [2], which, as the authors say, was significantly inspired by the fundamental research of Ivashev-Musatov [3], is the most important reference for our main theorem. We need some notation. Let G be an arbitrary nondiscrete locally compact Abelian group with the character group X. Let $M_s^+(G)$ be the set of positive singular measures on G, and let $d\chi$ be the Haar measure on X

Theorem of Hewitt and Ritter (Theorem 2.2(a) of [2])**.** *Let r be a real number greater than or equal to 2. There exists a measure $\theta \in M_s^+(G)$ with* *compact support such that $\theta(G) = 1$, the Fourier transform $\widehat{\theta}$ is a nonnegative continuous function on X tending to zero at infinity, and*

$$\int_X (\widehat{\theta}(\chi))^p \, d\chi$$

is finite if and only if $p > r$.

It is of interest that the proof of this theorem in [2] contains an explicit construction of the desired measure.

Let us now pass to our main result.

Theorem. *For every $\sigma \in (0, 2)$, the Hilbert space H_σ contains the Fourier transform of some singular continuous measure. Therefore, the space $\mathfrak{H}_\sigma$ contains the corresponding singular continuous measure.*

Proof. Let us apply the above Hewitt–Ritter theorem for $r = 2$ and find a measure $\theta \in M_s^+(\mathbb{C})$ with compact support such that $\theta(G) = 1$, the Fourier transform $\widehat{\theta}$ is a nonnegative continuous function on $\mathbb{C}$ tending to zero at infinity, and

$$\int_{\mathbb{C}} |\widehat{\theta}(w)|^p \, dw$$

is finite if and only if $p > 2$. Let us now consider the integral over the set $L = \mathbb{C} \setminus \{w : w \in \mathbb{C}, |w| < 1\}$. Certainly, for a given $\sigma \in (0, 2)$, the integral

$$\int_{\mathbb{C}} |\widehat{\theta}(w)|^2 |w|^{-\sigma} \, dw$$

is finite if and only if the integral

$$\int_L |\widehat{\theta}(w)|^2 |w|^{-\sigma} \, dw$$

is finite. Let us now use the Hölder inequality to estimate the integral

$$\int_L |\widehat{\theta}(w)|^2 |w|^{-\sigma} \, dw.$$

Take a $p > 2$; then, by the Hölder inequality,

$$\int_L |\widehat{\theta}(w)|^2 |w|^{-\sigma} \, dw \le \Big(\int_L |\widehat{\theta}(w)|^p \, dw \Big)^{2/p} \Big(\int_L |w|^{-q\sigma} \, dw \Big)^{1/q}, \tag{2}$$

where $2/p + 1/q = 1$, $q = \frac{p}{p-2}$. Let us choose the positive number $p - 2$ so small that, for a given $\sigma > 0$, the integral

$$\int_L |w|^{-\sigma \frac{p}{p-2}}\, dw$$

converges (it is sufficient to ensure that $\sigma \frac{p}{p-2} > 2$, i.e., $2 < p < \frac{4}{2-\sigma}$). Then the above Hölder inequality shows that the integral

$$\int_L |\widehat{\theta}(w)|^2 |w|^{-\sigma}\, dw$$

converges indeed for every $\sigma \in (0, 2)$, because the first integral on the right-hand side of (2) converges for $p > 2$ by the Hewitt–Ritter theorem. This completes the proof.

§ 3. Discussion

In fact, for $\sigma \in (1, 2)$, there is a very simple example of a singular continuous measure in the space $\mathfrak{H}_\sigma$, namely, the measure $\delta(|w| - a)$ uniformly distributed on a circle centered at the origin of positive radius a. Indeed, as is well known, the Fourier transform of this measure is equal to $2aJ_0(a|w|)$. Since

$$|J_0(a|w|)| \le \frac{const}{|w|^{1/2}}$$

at infinity, it follows that the integral

$$\int_{\mathbb{C}} |J_0(a|w|)|^2 |w|^{-\sigma}\, dw$$

converges for $\sigma \in (1, 2)$. Thus, the measure $\delta(|w| - a)$ belongs to the representation space $\mathfrak{H}_\sigma$ for $\sigma \in (1, 2)$.

§ 4. A question

The following questions arise in connection with the result of the present paper.

(1) Are there elements of representation spaces of the complementary series of representations of the group $\mathrm{SL}(2, \mathbb{C})$ that are not sums of ordinary functions and singular measures?

(2) Is it true that the family of bounded measurable functions $f \in H_\sigma$, $\sigma \in (0, 2)$, is contained in the space of Fourier transforms of the bounded regular Radon measures on $\mathbb{C}$?

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Department of Mechanics and Mathematics,
Moscow State University,
Moscow, 119991 Russia, and
Institute of Systems Research (VNIISI),
Russian Academy of Sciences,
Moscow, 117312 Russia
E-mail: ashtern@member.ams.org